# On the Error Probability of Optimal Codes in Gaussian Channels under Maximal Power Constraint

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***Abstract*—For an additive white Gaussian noise channel, we prove that Th. 41 in [Polyanskiy, Poor, Verdú 2010] is a lower bound to the error probability of any channel code satisfying the maximal power constraint. In contrast, the (tighter) lower bound to the error probability in Eq. (20) in [Shannon 1959] only holds under equal power constraint.**

## I. Introduction

We consider the problem of transmitting $M$ equiprobable messages over $n$ uses of an additive white Gaussian noise (AWGN) channel. In [1], Shannon derived a lower bound on the error probability for codes subject to a certain power constraint $\Gamma$. Using geometrical arguments, Shannon lower-bounded the error probability of a code with all the codewords lying on the $n$-dimensional sphere with squared radius $n\Gamma$ (*equal power constraint*) [1, Eq. (20)]. Then, he considered a length-$n$ code such that the codeword energy is not larger than $n\Gamma$ (*maximal power constraint*). He argued that such code can be transformed by adding an extra $(n+1)$-th coordinate to equalize the codeword energy to $n\Gamma$. As a result, the lower bound in [1, Eq. (20)], evaluated for the blocklength $n+1$, also holds for any length-$n$ maximal power constrained code.

More recently, Polyanskiy, Poor and Verdú proved that a surrogate binary hypothesis test can be used to lower bound the error probability of a channel code [2, Th. 27]. Particularizing this bound for the additive white Gaussian noise (AWGN) channel under equal power constraint yields [2, Th. 41]. As discussed above, evaluating [2, Th. 41] for a blocklength $n+1$ yields a converse bound for a length-$n$ code in the maximal power constraint setting.

While most of the analysis in [1] is focused in characterizing the asymptotics of [1, Eq. (20)], this bound is extremely accurate in the finite-length setting [3]. Indeed, in general, Shannon's approach yields tighter bounds than [2, Th. 41] under equal power constraint. In this work, we prove that [2, Th. 41] is directly a lower bound to the error probability of a length-$n$ maximal power constrained code (with no $n+1$ extension required). In contrast, Shannon lower bound only holds under equal power constraint, and the $n+1$ extension argument is needed in the maximal power constraint setting.

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## II. System Model and Preliminaries

We consider the problem of transmitting $M$ equiprobable messages over $n$ uses of an AWGN channel $W$ with noise power $\sigma^2$. Specifically, for the input $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and output $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ the channel $W = P_{\boldsymbol{Y}|\boldsymbol{X}}$ has a probability density function (pdf) given by

$$w(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^{n} \varphi_{x_i,\sigma}(y_i), \tag{1}$$

where $\varphi_{\mu,\sigma}(\cdot)$ denotes the pdf of the Gaussian distribution,

$$\varphi_{\mu,\sigma}(x) \triangleq \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{2}$$

The encoder maps a message $v \in \{1, \ldots, M\}$ to the channel as $\boldsymbol{x} = \boldsymbol{c}_v$ using the codebook $\mathcal{C} \triangleq \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_M\}$. Based on the channel output $\boldsymbol{y}$, the decoder guesses the transmitted message $\hat{v} \in \{1, \ldots, M\}$. The error probability is thus given by $P_{\mathrm{e}}(\mathcal{C}) \triangleq \Pr\{\hat{V} \neq V\}$ where the underlying probability is induced by the chain of source, encoder, channel and decoder. We consider codebooks satisfying a certain power constraint:

- Equal-power constrained codes,

$$\mathcal{L}_{\mathrm{e}}(\Gamma) \triangleq \Big\{\mathcal{C} \;\Big|\; \|\boldsymbol{c}_i\|^2 = n\Gamma, \quad i = 1, \ldots, M\Big\}. \tag{3}$$

- Maximal-power constrained codes,

$$\mathcal{L}_{\mathrm{m}}(\Gamma) \triangleq \Big\{\mathcal{C} \;\Big|\; \|\boldsymbol{c}_i\|^2 \leq n\Gamma, \quad i = 1, \ldots, M\Big\}. \tag{4}$$

- Average-power constrained codes,

$$\mathcal{L}_{\mathrm{a}}(\Gamma) \triangleq \Big\{\mathcal{C} \;\Big|\; \tfrac{1}{M}\sum_{i=1}^{M} \|\boldsymbol{c}_i\|^2 \leq n\Gamma\Big\}. \tag{5}$$

Clearly, $\mathcal{L}_{\mathrm{e}}(\Gamma) \subset \mathcal{L}_{\mathrm{m}}(\Gamma) \subset \mathcal{L}_{\mathrm{a}}(\Gamma)$. While the equal-power constraint is easier to analyze, the maximal and average-power constraints are more useful in practice. Here, we present lower bounds on $P_{\mathrm{e}}(\mathcal{C})$ under equal and maximal-power constraints.

### A. Shannon'59 lower bound

Let $\theta$ be the half-angle of a $n$-dimensional cone with vertex at the origin and with axis going through the vector $\boldsymbol{x} = (1, \ldots, 1)$. We denote by $\Phi_n(\theta, \sigma^2)$ the probability that such vector is moved outside this cone by effect of the i.i.d. Gaussian noise with variance $\sigma^2$ in each dimension.

*Theorem 1 ([1, Eq. (33)]):* Let $\mathcal{C} \in \mathcal{L}_{\mathrm{e}}(\Gamma)$ be a length-$n$ code of cardinality $M$ satisfying an equal power constraint.

Let $\theta_{n,M}$ denote the half-angle of a cone with solid angle equal to $\Omega_n/M$, where $\Omega_n$ is the surface of the $n$-dimensional hypersphere. Then,

$$P_{\rm e}(\mathcal{C}) \geq \Phi_n\left(\theta_{n,M}, \frac{\sigma^2}{\Gamma}\right). \tag{6}$$

While this bound is conceptually simple and accurate for relatively short codes [3], it is difficult to evaluate. The computation of this bound is treated, e.g., in [4], [5].

### *B. PPV'10 lower bound*

In [2], Polyanskiy *et al.* proved that the error probability of a binary hypothesis test with certain parameters can be used to lower bound the error probability $P_{\rm e}(\mathcal{C})$ for a certain channel $P_{\boldsymbol{Y}|\boldsymbol{X}}$. In particular, [2, Th. 27] shows that

$$P_{\rm e}(\mathcal{C}) \geq \inf_{P_{\boldsymbol{X}}} \sup_{Q_{\boldsymbol{Y}}} \left\{ \alpha_{\frac{1}{M}}\left(P_{\boldsymbol{X}}P_{\boldsymbol{Y}|\boldsymbol{X}}, P_{\boldsymbol{X}} \times Q_{\boldsymbol{Y}}\right)\right\}, \tag{7}$$

where $\alpha_\beta\left(P,Q\right)$ is the minimum type-I error for a maximum type-II error $\beta \in [0,1]$ in a binary hypothesis testing problem between the distributions $P$ and $Q$.

The bound (7) is usually referred to as the *meta-converse bound* since several converse bounds in the literature can be recovered from it via relaxation. While it is possible to restrict the set of distributions $Q_{\boldsymbol{Y}}$ over which the bound is maximized and still obtain a lower bound, the minimization over $P_{\boldsymbol{X}}$ needs to be carried out over all the $n$-dimensional probability distributions (not necessarily product) satisfying the power constraint considered.

For the Gaussian channel, Polyanskiy *et al.* fixed $Q_{\boldsymbol{Y}}$ to be zero-mean Gaussian distributed with variance $\theta^2$ and independent entries, i.e., $Q_{\boldsymbol{Y}} = Q$ with pdf

$$q(\boldsymbol{y}) = \prod_{i=1}^{n} \varphi_{0,\theta}(y_i). \tag{8}$$

Particularizing (7) for this channel and fixing $Q_{\boldsymbol{Y}} = Q$, yields

$$P_{\rm e}(\mathcal{C}) \geq \inf_{P \in \mathcal{P}_\Gamma} \left\{ \alpha_{\frac{1}{M}}\left(PW, P \times Q\right)\right\}, \tag{9}$$

where the minimization is over all input distributions $P$ satisfying a certain power constraint $\Gamma$, denoted by $\mathcal{P}_\Gamma$. For this choice of $Q$, $\alpha_{\frac{1}{M}}(\cdot,\cdot)$ presents spherical symmetry. Then, restricting the input codebook to lie on the surface of a $n$-dimensional hyper-sphere of squared radius $n\Gamma$ (equal power constraint), setting $\theta^2 = \Gamma + \sigma^2$, the following result follows.

*Theorem 2 ([2, Th. 41]):* Let $\mathcal{C} \in \mathcal{L}_{\rm e}(\Gamma)$ be a length-$n$ code of cardinality $M$ satisfying an equal power constraint. Then,

$$P_{\rm e}(\mathcal{C}) \geq \alpha_{\frac{1}{M}}\left(\varphi^n_{\sqrt{\Gamma},\sigma}, \varphi^n_{0,\theta}\right), \tag{10}$$

where $\theta^2 = \Gamma + \sigma^2$.

This expression can be evaluated via the probability of two noncentral $\chi^2$ distributions (see Appendix A for details). However, for fixed rate $R \triangleq \frac{1}{n}\log_2 M$, the term $\frac{1}{M} = 2^{-nR}$ decreases exponentially with the block-length and traditional series series expansions of the noncentral $\chi^2$ fail even for moderate values of $n$ (see discussion in [2, p. 2326]).

(a) (b)

Fig. 1: Induced integration regions by (a) the Shannon'59 lower bound (6), and (b) the PPV'10 lower bound (10).

### *C. Comparison between Shannon'59 and PPV'10*

Shannon'59 lower bound in Theorem 1 corresponds to the probability that the additive Gaussian noise moves a given codeword out of the $n$-dimensional cone centered at the codeword (cone that roughly covers $1/M$-th of the output space). We show next that the PPV'10 lower bound in Theorem 2 admits an analogous geometrical interpretation.

Let $\boldsymbol{x}$ satisfy $\|\boldsymbol{x}\|^2 = n\Gamma$ and $\theta^2 = \Gamma + \sigma^2$. For the hypothesis test on the right-hand side of (10), the condition

$$\frac{\varphi^n_{\rho,\sigma}(\boldsymbol{y})}{\varphi^n_{0,\theta}(\boldsymbol{y})} = \frac{\theta^n}{\sigma^n} \exp\left[\frac{\|\boldsymbol{y}\|^2}{2\theta^2} - \frac{\|\boldsymbol{y}-\boldsymbol{x}\|^2}{2\sigma^2}\right] = \gamma \tag{11}$$

for some $\gamma > 0$, defines the boundary of the decision region induced by the optimal Neyman-Pearson test. We next study the shape of this region. To this end, we note that

$$\frac{\|\boldsymbol{y}\|^2}{2\theta^2} - \frac{\|\boldsymbol{y}-\boldsymbol{x}\|^2}{2\sigma^2} = -\frac{\theta^2-\sigma^2}{2\sigma^2\theta^2}\left(\|\boldsymbol{y}\|^2 - 2a\langle\boldsymbol{x},\boldsymbol{y}\rangle + a\|\boldsymbol{x}\|^2\right) \tag{12}$$

$$= -\frac{\theta^2-\sigma^2}{2\sigma^2\theta^2}\left(\|\boldsymbol{y}-a\boldsymbol{x}\|^2 + (a-a^2)\|\boldsymbol{x}\|^2\right), \tag{13}$$

where $a = \frac{\theta^2}{\theta^2-\sigma^2} \geq 0$ for $\theta^2 \geq \sigma^2$, and where $\langle\boldsymbol{x},\boldsymbol{y}\rangle$ denotes the inner product between $\boldsymbol{x}$ and $\boldsymbol{y}$.

Using (13) with $\|\boldsymbol{x}\|^2 = n\Gamma$ and $\theta^2 = \Gamma + \sigma^2$, we obtain that the boundary of the decision region (11) becomes

$$\left\|\boldsymbol{y} - \left(1 + \tfrac{\sigma^2}{\Gamma}\right)\boldsymbol{x}\right\|^2 = \bar{\gamma}, \tag{14}$$

where $\bar{\gamma} = n\sigma^2\left(1+\frac{\sigma^2}{\Gamma}\right)\left(1 + \log\left(1+\frac{\Gamma}{\sigma^2}\right) + \frac{2}{n}\log(\gamma)\right)$.

As (14) corresponds to the equation of an $n$-dimensional sphere, we can alternatively describe the PPV'10 lower bound in Theorem 2 as the probability that the additive Gaussian noise moves the codeword $\boldsymbol{x}$ out of the $n$-dimensional sphere centered at $\left(1+\frac{\sigma^2}{\Gamma}\right)\boldsymbol{x}$ (that covers $1/M$-th of the output space). Note that the "regions" induced by Theorem 1 correspond to cones, while those induced by Theorem 2 correspond to spheres (see Fig. 1). Cones are close to the optimal ML decoding regions for codewords evenly distributed on surface of an $n$-dimensional sphere with squared radius $n\Gamma$.[1] On the other hand, "spherical regions" allow different configurations of the codewords inside the sphere. Then, the meta-converse bound may hold beyond the equal-power constraint.

This intuition is proven to be right in the next section.

[1]Indeed, in $n = 2$ dimensions Shannon'59 lower bound yields the exact error probability of an $M$-PSK constellation. See Section III-A for details.

## III. Lower Bound for Maximal-power Constraints

In order to lower bound the error probability of a maximal-power constrained codebook we start by considering the general meta-converse in (7). In order to make the minimization over $P_{\boldsymbol{X}}$ in (7) tractable we shall use the following result.

*Lemma 1 ([6, Lem. 25]):* Let $P_{\boldsymbol{X}} = \sum_j \lambda_j P_{\boldsymbol{X}_j}$ with $\lambda_j > 0$, $\sum_j \lambda_j = 1$, be a convex combination of the distributions $P_{\boldsymbol{X}_j}$ and let $\{P_{\boldsymbol{X}_j}\}$ have pairwise disjoint supports. Then, the hypothesis testing error trade-off function satisfies

$$\alpha_\beta\bigl(P_{\boldsymbol{X}} P_{\boldsymbol{Y}|\boldsymbol{X}}, P_{\boldsymbol{X}} \times Q_{\boldsymbol{Y}}\bigr) = \min_{\substack{\{\beta_j\}:\\ \beta=\sum_j \lambda_j\beta_j}} \sum_j \lambda_j \alpha_{\beta_j}\bigl(P_{\boldsymbol{X}_j} P_{\boldsymbol{Y}|\boldsymbol{X}}, P_{\boldsymbol{X}_j} \times Q_{\boldsymbol{Y}}\bigr). \quad (15)$$

This lemma asserts that it is possible to express the test (7) as a convex combination of disjoint sub-tests provided that the type-II error is optimally distributed among them. Applying this decomposition in (9) for the Gaussian channel under maximal power constraint, we obtain the following result.

*Theorem 3 (Maximal power constraint):* Let $\mathcal{C} \in \mathcal{L}_{\mathrm{m}}(\Gamma)$ be a length-$n$ code of cardinality $M$ satisfying a maximal power constraint and let $n \geq 1$. Then, for any $\theta > \sigma$,

$$P_{\mathrm{e}}(\mathcal{C}) \geq \alpha_{\frac{1}{M}}\bigl(\varphi^n_{\sqrt{\Gamma},\sigma}, \varphi^n_{0,\theta}\bigr). \quad (16)$$

*Proof:* For any $0 \leq \rho \leq \sqrt{\Gamma}$, we define the input set $\mathcal{S}_\rho \triangleq \bigl\{\boldsymbol{x} \,|\, \|\boldsymbol{x}\|^2 = n\rho^2\bigr\}$. Then, any input distribution $P_{\boldsymbol{X}}$ induces a distribution over the parameter $\rho$, $P_\rho \triangleq \Pr\{\mathcal{S}_\rho\}$. We consider the conditional distribution

$$\mathrm{d}P_{\boldsymbol{X}|\rho}(\boldsymbol{x}) = \begin{cases} \frac{\mathrm{d}P_{\boldsymbol{X}}(\boldsymbol{x})}{\mathrm{d}P_\rho}, & \boldsymbol{x} \in \mathcal{S}_\rho,\\ 0, & \text{otherwise.}\end{cases} \quad (17)$$

It follows that $P_{\boldsymbol{X}}(\boldsymbol{x}) = \int P_{\boldsymbol{X}|\rho}(\boldsymbol{x})\,\mathrm{d}P_\rho$ with $\mathrm{d}P_\rho$ satisfying $\mathrm{d}P_\rho \geq 0$, $\int \mathrm{d}P_\rho = 1$. Then, we apply Lemma 1 to the right-hand side of (9) to obtain

$$\inf_{P\in\mathcal{P}_\Gamma} \Bigl\{\alpha_{\frac{1}{M}}\bigl(PW, P\times Q\bigr)\Bigr\} = \inf_{\substack{\{P_\rho,\beta_\rho\}:\\ \int \beta_\rho\,\mathrm{d}P_\rho = \frac{1}{M}}} \left\{\int \alpha_{\beta_\rho}\bigl(P_\rho W, P_\rho \times Q\bigr)\,\mathrm{d}P_\rho\right\} \quad (18)$$

$$= \inf_{\substack{\{P_\rho,\beta_\rho\}:\\ \int \beta_\rho\,\mathrm{d}P_\rho = \frac{1}{M}}} \left\{\int \alpha_{\beta_\rho}\bigl(\varphi^n_{\rho,\sigma}, \varphi^n_{0,\theta}\bigr)\,\mathrm{d}P_\rho\right\}, \quad (19)$$

where the last step follows from the spherical symmetry of each of the sub-tests in (18) and since $\boldsymbol{x} = (\rho,\ldots,\rho) \in \mathcal{S}_\rho$.

To solve the optimization in (19) we resort in the following lemma, which is then proven in the appendices.

*Lemma 2:* Let $\sigma < \theta$, with $\sigma,\theta \in \mathbb{R}^+$ and $n \geq 1$. Then, $\alpha_\beta\bigl(\varphi^n_{\rho,\sigma}, \varphi^n_{0,\theta}\bigr)$ is non-increasing in $\rho$ for any fixed $\beta \in [0,1]$.

According to Lemma 2, for any $0 \leq \rho \leq \sqrt{\Gamma}$, it holds that $\alpha_\beta\bigl(\varphi^n_{\rho,\sigma}, \varphi^n_{0,\theta}\bigr) \geq \alpha_\beta\bigl(\varphi^n_{\sqrt{\Gamma},\sigma}, \varphi^n_{0,\theta}\bigr)$. As any maximal-power constrained input distribution $P \in \mathcal{P}_\Gamma$ satisfies $P_\rho = 0$ for $\rho > \sqrt{\Gamma}$, we conclude that

$$\inf_{\substack{\{P_\rho,\beta_\rho\}:\\ \int \beta_\rho\,\mathrm{d}P_\rho = \frac{1}{M}}} \left\{\int \alpha_{\beta_\rho}\bigl(\varphi^n_{\rho,\sigma}, \varphi^n_{0,\theta}\bigr)\,\mathrm{d}P_\rho\right\}$$

$$\geq \inf_{\substack{\{P_\rho,\beta_\rho\}:\\ \int \beta_\rho\,\mathrm{d}P_\rho = \frac{1}{M}}} \left\{\int \alpha_{\beta_\rho}\bigl(\varphi^n_{\sqrt{\Gamma},\sigma}, \varphi^n_{0,\theta}\bigr)\,\mathrm{d}P_\rho\right\} \quad (20)$$

$$\geq \alpha_{\frac{1}{M}}\bigl(\varphi^n_{\sqrt{\Gamma},\sigma}, \varphi^n_{0,\theta}\bigr), \quad (21)$$

where in (21) we used that the function $\alpha_\beta(\cdot,\cdot)$ is convex with respect to $\beta$, hence, $\int \alpha_{\beta_\rho}(\cdot,\cdot)\,\mathrm{d}P_\rho \geq \alpha_{\int \beta_\rho\,\mathrm{d}P_\rho}(\cdot,\cdot)$.

Then, using (9), (19) and (21) the result follows. ■

Setting $\theta^2 = \Gamma + \sigma^2$ in Theorem 3, we recover the bound in Theorem 2. We conclude that the bound in Theorem 2 also holds for maximal power constraint. This is not the case however for the Shannon'59 lower bound in Theorem 1, as we show next with an example.

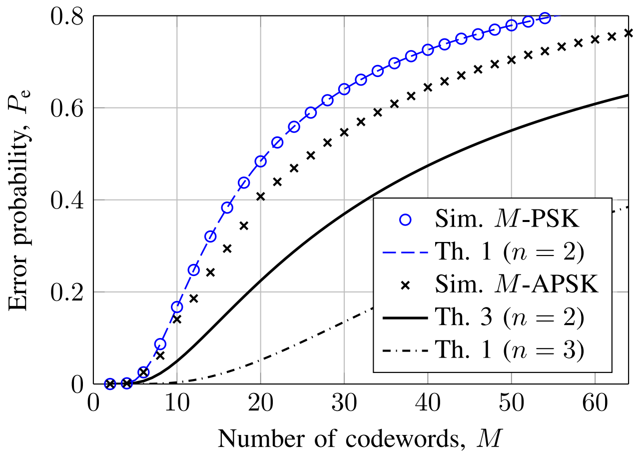


Fig. 2: Lower bounds to the channel coding error probability over an AWGN channel with $n = 2$ and SNR= 10 dB.

### A. *Example: 2-dimensional constellations*

We consider the problem of transmitting $M \geq 2$ codewords over a additive Gaussian noise channel with $n = 2$ dimensions. Figure 2 compares the bounds in Theorem 1 (evaluated for $n = 2$ and $n = 3$) and Theorem 3 with $\theta^2 = \Gamma + \sigma^2$. For reference, we include the simulated ML decoding error probability of an $M$-PSK (phase-shift keying) and $M$-APSK (amplitude-phase-shift keying) constellations satisfying the maximal power constraint. For $n = 2$, Shannon'59 lower bound in Theorem 1 coincides with the ML decoding error probability of the $M$-PSK constellation (as the 2-dimensional cones are precisely the ML decoding regions of the $M$-PSK constellation). Theorem 1 only applies for codebooks (or constellations) satisfying the equal power constraint. Indeed, the $M$-APSK simulated error probability violates the bound evaluated for $n = 2$. Theorem 3 applies to both equal and maximal power constraints, as it does Theorem 1 evaluated for $n = 3$. We can see that Theorem 3 is tighter in this setting.

## APPENDIX A
## PROOF OF LEMMA 2

Let $\sigma, \theta > 0$ and $n \geq 1$, be fixed parameters. We define

$$\jmath_\rho(\boldsymbol{y}) \triangleq \log \frac{\varphi^n_{\rho,\sigma}(\boldsymbol{y})}{\varphi^n_{0,\theta}(\boldsymbol{y})} \tag{22}$$

$$= \log\frac{\theta}{\sigma} + \frac{1}{2}\sum_{i=1}^{n} \frac{\theta^2(y_i-\rho)^2 - \sigma^2 y_i^2}{\sigma^2\theta^2}. \tag{23}$$

The trade-off $\alpha_\beta\big(\varphi^n_{\rho,\sigma}, \varphi^n_{0,\theta}\big)$ admits the parametric form

$$\alpha(\rho,\gamma) = \Pr\big[\jmath_\rho(\boldsymbol{Y}_0) \leq \gamma\big] = \Pr\big[\jmath_{0,\rho}(\boldsymbol{Z}) \leq \gamma\big], \tag{24}$$

$$\beta(\rho,\gamma) = \Pr\big[\jmath_\rho(\boldsymbol{Y}_1) > \gamma\big] = \Pr\big[\jmath_{1,\rho}(\boldsymbol{Z}) > \gamma\big], \tag{25}$$

in terms of the auxiliary parameter $\gamma \in \mathbb{R}$. Here, $\boldsymbol{Y}_0 \sim \varphi^n_{\rho,\sigma}$, $\boldsymbol{Y}_1 \sim \varphi^n_{0,\theta}$ and, for $\boldsymbol{Z} \sim \varphi^n_{0,1}$ and $\delta \triangleq \theta^2 - \sigma^2$, we defined

$$\jmath_{0,\rho}(\boldsymbol{z}) \triangleq \log\frac{\theta}{\sigma} - \frac{n}{2}\frac{\rho^2}{\delta} + \frac{1}{2}\frac{\delta}{\sigma^2}\sum_{i=1}^{n}\Big(z_i - \frac{\sigma\rho}{\delta}\Big)^2, \tag{26}$$

$$\jmath_{1,\rho}(\boldsymbol{z}) \triangleq \log\frac{\theta}{\sigma} - \frac{n}{2}\frac{\rho^2}{\delta} + \frac{1}{2}\frac{\delta}{\theta^2}\sum_{i=1}^{n}\Big(z_i - \frac{\theta\rho}{\delta}\Big)^2. \tag{27}$$

The equivalence between the 1st and 2nd identities in (24) and (25) follows from (23), (26) and (27) via a change of variables.

Given (26) and (27), since $\boldsymbol{Z} \sim \varphi^n_{0,1}$, we conclude that $\jmath_{0,\rho}(\boldsymbol{Z})$ and $\jmath_{1,\rho}(\boldsymbol{Z})$ follow a (shifted and scaled) noncentral $\chi^2$ distribution with $n$ degrees of freedom and non-centrality parameters $n\sigma^2\rho^2/\delta^2$ and $n\theta^2\rho^2/\delta^2$, respectively. The cdf of a noncentral $\chi^2$ distribution can be written in terms of the generalized Marcum $Q$-function $Q_m(a,b)$ defined in (37). Then, using (24), (25), (26) and (27), we characterize $\alpha_\beta\big(\varphi^n_{\rho,\sigma}, \varphi^n_{0,\theta}\big)$ as a function of an auxiliary parameter $\tilde{\gamma} \geq 0$ as

$$\alpha(\rho,\tilde{\gamma}) = Q_{\frac{n}{2}}\left(\sqrt{n}\frac{\sigma\rho}{\delta}, \frac{\tilde{\gamma}}{\sigma}\right), \tag{28}$$

$$\beta(\rho,\tilde{\gamma}) = 1 - Q_{\frac{n}{2}}\left(\sqrt{n}\frac{\theta\rho}{\delta}, \frac{\tilde{\gamma}}{\theta}\right). \tag{29}$$

To prove that $\alpha_\beta\big(\varphi^n_{\rho,\sigma}, \varphi^n_{0,\theta}\big)$ is non-increasing in $\rho$, we need to show that its derivative with respect to $\rho$ is non-positive. To this end, we could invert (29) to obtain the dependence of $\tilde{\gamma}$ with $\rho$ for fixed $\beta$ and substitute this $\tilde{\gamma}(\rho)$ in (28) before taking the derivative. However, given the nature of the functions involved, there is no closed-form expression for $\tilde{\gamma}(\rho)$. Instead, we use the chain rule for total derivatives to write

$$\frac{\partial\beta(\rho,\tilde{\gamma})}{\partial\rho} = \frac{\partial\beta(\rho,\tilde{\gamma})}{\partial\rho} + \frac{\partial\beta(\rho,\tilde{\gamma})}{\partial\tilde{\gamma}}\frac{\partial\tilde{\gamma}}{\partial\rho}. \tag{30}$$

As $\beta$ is fixed, we set (30) equal to 0 and solve for $\frac{\partial\tilde{\gamma}}{\partial\rho}$. Then,

$$\frac{\partial\tilde{\gamma}}{\partial\rho} = -\frac{\frac{\partial}{\partial\rho}\beta(\rho,\tilde{\gamma})}{\frac{\partial}{\partial\tilde{\gamma}}\beta(\rho,\tilde{\gamma})} = \frac{I_{\frac{n}{2}}\big(\sqrt{n}\frac{\rho\tilde{\gamma}}{\delta}\big)\sqrt{n}\frac{\theta}{\delta}}{I_{\frac{n}{2}-1}\big(\sqrt{n}\frac{\rho\tilde{\gamma}}{\delta}\big)\frac{1}{\theta}}, \tag{31}$$

where $I_m(\cdot)$ is the $m$-th order modified Bessel function of the first kind and where we used that (see Appendix B)

$$\frac{\partial Q_m(a,b)}{\partial a} = \frac{b^m}{a^{m-1}} e^{-\frac{a^2+b^2}{2}} I_m(ab), \tag{32}$$

$$\frac{\partial Q_m(a,b)}{\partial b} = -\frac{b^m}{a^{m-1}} e^{-\frac{a^2+b^2}{2}} I_{m-1}(ab). \tag{33}$$

We now evaluate the derivative of $\frac{\partial\alpha}{\partial\rho}$ for fixed $\beta$. By applying the chain rule for total derivatives and using (31), (32) and (33), we obtain

$$\frac{\partial\alpha(\rho,\tilde{\gamma})}{\partial\rho} = \frac{\partial\alpha(\rho,\tilde{\gamma})}{\partial\rho} + \frac{\partial\alpha(\rho,\tilde{\gamma})}{\partial\tilde{\gamma}}\frac{\partial\tilde{\gamma}}{\partial\rho} \tag{34}$$

$$= -\frac{\sqrt{n}}{\sigma}\frac{b^{\frac{n}{2}}}{a^{\frac{n}{2}-1}} e^{-\frac{a^2+b^2}{2}} I_{\frac{n}{2}}\big(\sqrt{n}\tfrac{\rho\tilde{\gamma}}{\delta}\big) \tag{35}$$

$$= -\frac{n\rho}{\delta}\Big(\frac{\tilde{\gamma}\delta}{\sqrt{n}\sigma^2\rho}\Big)^{\frac{n}{2}} e^{-\frac{n\sigma^4\rho^2+\delta^2\tilde{\gamma}^2}{2\delta^2\sigma^2}} I_{\frac{n}{2}}\big(\sqrt{n}\tfrac{\rho\tilde{\gamma}}{\delta}\big) \tag{36}$$

where $a = \sqrt{n}\frac{\sigma\rho}{\delta}$ and $b = \frac{\tilde{\gamma}}{\sigma}$ in (35). As (36) is non-positive for $\delta = \theta^2 - \sigma^2 > 0$, then Lemma 2 follows.

## APPENDIX B
## DERIVATIVES OF THE MARCUM-$Q$ FUNCTION

For $a > 0$ and $b > 0$, the Marcum-$Q$ function is defined as

$$Q_m(a,b) \triangleq \int_b^\infty \frac{t^m}{a^{m-1}} e^{-\frac{t^2+a^2}{2}} I_{m-1}(at)\,\mathrm{d}t. \tag{37}$$

The derivative (33) then follows directly from (37). For (32) we make use of the series representation [7, Eq. (4.62)]

$$Q_m(a,b) = e^{-\frac{t^2+a^2}{2}} \sum_{r=1-m}^{\infty} \Big(\frac{a}{b}\Big)^r I_{-r}(ab) \tag{38}$$

and we write its derivative with respect to $a$ to obtain

$$\frac{\partial Q_m(a,b)}{\partial a} = e^{-\frac{t^2+a^2}{2}} \sum_{1-m}^{\infty} \Big(\frac{a}{b}\Big)^r \Big(\Big(\frac{r}{a} - a\Big) I_{-r}(ab) + b I'_{-r}(ab)\Big). \tag{39}$$

Using the identity $I'_m(x) = \frac{m}{x} I_m(x) + I_{m+1}(x)$ [8, Sec. 8.486] and canceling terms we obtain (32). To the best of our knowledge, the form of the derivative in (32) does not appear in the literature for non-integer values of $m$. For integer values of $m$, (32) can be easily obtained from (37) by using the identities $Q_m(a,b) = 1 - Q_{1-m}(b,a)$ and $I_m(x) = I_{-m}(x)$.

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## REFERENCES


[1] C. Shannon, "Probability of error for optimal codes in a Gaussian channel," *Bell System Technical Journal*, vol. 38, p. 611656, 1959.

[2] Y. Polyanskiy, H. V. Poor, and S. Verdú, "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2307–2359, 2010.

[3] I. Sason and S. Shamai (Shitz), *Performance analysis of linear codes under maximum-likelihood decoding: a tutorial*. Foundations and Trends Commun. and Inf. Theory, now Publishers, 2006.

[4] A. Valembois and M. Fossorier, "Sphere-packing bounds revisited for moderate block lengths," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 2998–3014, 2004.

[5] G. Wiechman and I. Sason, "An improved sphere-packing bound for finite-length codes over symmetric memoryless channels," *IEEE Trans. Inf. Theory*, vol. 54, no. 5, pp. 1962–1990, May 2008.

[6] Y. Polyanskiy, "Saddle point in the minimax converse for channel coding," *IEEE Trans. Inf. Theory*, vol. 59, no. 5, pp. 2576–2595, May 2013.

[7] M.-S. A. Marvin K. Simon, *Digital Communication over Fading Channels*, 2nd ed. New Jersey: Wiley-IEEE Press, 2004.

[8] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. London: Elsevier, 2007.